JOURNAL OF COMPUTATIONAL PHYSICS 111, 399–409 (1994)

The Pseudospectral Method for Solving Differential Eigenvalue Problems

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Received October 17, 1991

A simple pseudospectral method is presented for the numerical solution of linear, differential eigenvalue problems. The method does not produce the spurious eigenvalues which generally occur when such problems are solved by the spectral tau method. The pseudospectral method is presented using two model problems, and the presentation contains a useful algorithm for the computation of the spectral differentiation matrices at general collocation points. Numerical results are offered for a set of benchmark problems, including the Orr–Sommerfeld equation for stability of plane Poiseuille flow between parallel plates. The results indicate that the new pseudospectral method is comparable in accuracy to the tau method. © 1994 Academic Press, Inc.

1. INTRODUCTION

Spectral methods offer viable alternatives to finite difference and finite element methods for solving differential equations and differential eigenvalue problems [1]. Since Orszag's influential paper [6] spectral methods—particularly of the spectral tau variety—have proven to be efficient techniques for providing accurate solutions to eigenvalue problems which arise in hydrodynamic stability calculations. However, a disadvantage of the spectral tau method is that it produces spurious (unstable) eigenvalues which are consequences of the method of discretisation. The occurrence of spurious eigenvalues has been discussed by many authors, and the reader is referred to the recent papers by Gardner *et al.* [2] and McFadden *et al.* [5] and to references therein.

It is well known that among spectral methods the pseudospectral approach is especially attractive, owing to the ease with which it can be applied to variable-coefficient problems and nonlinear problems. For eigenvalue problems the pseudospectral approximation is less straightforward than the spectral tau approximation if the interpolating polynomial is not chosen properly. Canuto *et al.* [1] illustrate the difficulties by discussing pseudospectral approximations to Orr–Sommerfeld equations. Of the three approximations which they present, only the one devised by Herbert [3] has been used in hydrodynamic stability calculations. Herbert found his approximation to be comparable in accuracy to the spectral tau approximation, but it produces spurious eigenvalues.

The objective of this paper is to present an extremely simple pseudospectral method for eigenvalue problems with Dirichlet boundary conditions (also called damped boundary conditions in plate deflection theory). Several numerical examples are presented which show that the method produces no spurious eigenvalues and that it is comparable in accuracy to the spectral tau method.

In Section 2 of the paper we describe the method using two model eigenvalue problems. Section 3 deals with the calculation of the pseudospectral differentiation matrices for general collocation points. Section 4 contains numerical results for several eigenvalue problems, including the Orr–Sommerfeld problem for plane Poiseuille flow. Conclusions and comments are presented in Section 5. An appendix is included which contains a simple preconditioner for pseudospectral approximations to nth-order differential operators in one dimension. This preconditioner is very effective in reducing the condition number of a pseudospectral differentiation matrix. Here it is used to scale the condition numbers of matrices produced by applying the pseudospectral method to eigenvalue problems.

* Supported by a Royal Fellowship Award from the Royal Society of London.

0021-9991/94 $6.00

2. A PSEUDOSPECTRAL METHOD FOR SOLVING EIGENVALUE PROBLEMS

In Gardner *et al.* [2] and McFadden *et al.* [5] the *standard spectral tau approximations to several problems* are observed to have spurious eigenvalues. Two of these problems are used in this section to describe our pseudospectral method.

Throughout this paper it is assumed that the independent variable x is in the interval $[-1, 1]$ and the collocation points are

$$x_1 \equiv -1 < x_2 < \cdots < x_{N-1} < x_N \equiv 1. \tag{2.1}$$

The symbol $P_n^{(\alpha,\beta)}$ denotes the Jacobi polynomial of degree n with parameters α and β. We denote the Lagrange interpolating polynomial for points (2.1) by $l_k(x)$, $1 \leqslant k \leqslant N$; that is,

$$l_k(x) = \prod_{\substack{i=1 \\ i \neq k}}^{N} \frac{x - x_i}{x_k - x_i}. \tag{2.2}$$

2.1. EXAMPLE 1. (*A fourth-order eigenvalue problem*). Consider the eigenvalue problem

$$u'''' + Ru''' = su'', \qquad -1 < x < 1, \tag{2.3}$$

$$u(-1) = u'(-1) = u(1) = u'(1) = 0, \tag{2.4}$$

where u is the unknown function, R is a real parameter, s is the eigenparameter, and a prime denotes differentiation with respect to x. Problem (2.3)–(2.4) is sufficiently simple to be solved analytically, while retaining the essential features necessary to illustrate the application of the pseudospectral method. The eigencondition for this problem is [2]

$$(R^2+4s)^{1/2}\left[1 - \frac{\cosh(R^2+4s)^{1/2}}{\cosh R}\right] + \frac{2s\sinh(R^2+4s)^{1/2}}{\cosh R} = 0. \tag{2.5}$$

To illustrate our pseudospectral approximation to this problem we make use of the two interpolation problems which are presented below. Details concerning the interpolation may be found in Huang and Sloan [4]:

(i) Find the polynomial u^N of degree $N+1$ such that

$$\begin{aligned} u^N(x_j) &= u(x_j), \qquad 2 \leqslant j \leqslant N-1, \\ u^N(\pm 1) &= u(\pm 1) = 0, \\ (u^N)'(\pm 1) &= u'(\pm 1) = 0. \end{aligned} \tag{2.6}$$

The solution of (2.6) is given by

$$u^N(x) = \sum_{k=2}^{N-1} u_k h_k(x), \tag{2.7}$$

where $u_k = u(x_k)$ and

$$h_k(x) = \frac{(1-x^2)}{(1-x_k^2)} l_k(x). \tag{2.8}$$

(ii) Find the polynomial $\tilde{u}^N$ of degree $N-1$ such that

$$\begin{aligned} \tilde{u}^N(x_j) &= u(x_j), \qquad 2 \leqslant j \leqslant N-1, \\ \tilde{u}^N(\pm 1) &= u(\pm 1) = 0. \end{aligned} \tag{2.9}$$

The solution of (2.9) is given by

$$\tilde{u}^N(x) = \sum_{k=2}^{N-1} u_k l_k(x), \tag{2.10}$$

It is worth noting that $u^N(x)$ is often used for fourth-order differential equations with boundary conditions $u(\pm 1) = u'(\pm 1) = 0$ and $\tilde{u}^N(x)$ is often used for second-order differential equations with boundary conditions $u(\pm 1) = 0$. See, for example, Huang and Sloan [4] and Canuto *et al.* [1].

Now the pseudospectral scheme is considered for (2.3)–(2.4). It is defined by the collocation equations

$$\frac{d^4u^N}{dx^4}(x_j) + R\frac{d^3u^N}{dx^3}(x_j) = s\frac{d^2\tilde{u}^N}{dx^2}(x_j), \qquad 2 \leqslant j \leqslant N-1. \tag{2.11}$$

The key feature of this scheme is that it uses two distinct interpolation polynomials. The degree of the polynomial which approximates the second-order derivative term on the right-hand side of (2.3) is lower by two than the degree of the polynomial which approximates the other terms in the equation. This approach has characteristics in common with the modified tau method of McFadden *et al.* [5]. In applying [5] to (2.3), u is approximated by a truncated Chebyshev series of degree N, with coefficients which we may denote by $a_0, a_1, ..., a_N$. Orthogonality properties of Chebyshev polynomials yield equations relating the Chebyshev coefficients of the terms in (2.3), and these coefficients are expressed as expansions in $a_0, a_1, ..., a_N$. The key feature in [5] is that the expansions of the Chebyshev coefficients of the second-order derivative term are modified by neglecting a_{N-1} and a_N: the second-order derivative term is therefore expressed in terms of the coefficients of a truncated Chebyshev series of degree $N-2$.

In the sense outlined above, Eq. (2.11) might be regarded as a pseudospectral analogue of Eq. (7a) in [5]. However, (2.11) seems to adopt a more natural viewpoint, since $\tilde{u}^N(x)$ is simply the standard pseudospectral approximation to the second-order derivative operator subject to the homogeneous conditions $\tilde{u}^N(\pm 1)=0$. The limiting solution as $|s| \to \infty$ is clearly $\tilde{u}^N(x) \equiv 0$.

We now return to Eq. (2.11). Let

$$\begin{aligned} A_{j-1,k-1} &= \frac{d^4 h_k}{dx^4}(x_j) + R\frac{d^3 h_k}{dx^3}(x_j), \\ B_{j-1,k-1} &= \frac{d^2 l_k}{dx^2}(x_j), \qquad 2 \leqslant k, j \leqslant N-1, \end{aligned} \tag{2.12}$$

$$\mathbf{u} = [u_2, ..., u_{N-1}]^{\mathrm{T}}. \tag{2.13}$$

Making use of (2.7), (2.8), and (2.10) we may write (2.11) as an $(N-2)\times(N-2)$ generalised matrix eigenvalue problem of the form

$$A\mathbf{u} = sB\mathbf{u}. \tag{2.14}$$

Equation (2.14) may be solved by any routine for generalised eigenvalue problems.

It should be noted that the condition numbers of matrices A and B—especially A—may be huge in some cases. For such cases, the solution of (2.14) will suffer from roundoff error, and overflow may even occur. A simple preconditioner, D, is described in the Appendix which permits (2.14) to be replaced by the scaled system

$$DA\mathbf{u} = sDB\mathbf{u}, \tag{2.15}$$

where DA has a much lower condition number than A. The scaled system (2.15) was used in all our calculations and it was found that D is extremely effective.

The calculation of the differentiation matrices A and B for the general collocation points x_j, $1 \leqslant j \leqslant N$, will be discussed in Section 3 of this paper. Numerical examples of the use of the pseudospectral method proposed here will be presented in Section 4.

2.2. Example 2. (A sixth-order eigenvalue problem). Another example used to illustrate our method is

$$\frac{d^6 u}{dx^6} = s\frac{d^4 u}{dx^4}, \qquad -1 < x < 1, \tag{2.16}$$

$$u(\pm 1) = u'(\pm 1) = u''(\pm 1) = 0. \tag{2.17}$$

It is shown in McFadden *et al.* [5] that the spectral tau discretisation of (2.16) and (2.17) has four spurious eigenvalues of which two are complex conjugate pairs with $\mathrm{Re}(s)<0$, but $\mathrm{Im}(s)\neq 0$. As in the case of Example 1 we illustrate the pseudospectral approximation using an interpolation problem.

Consider the polynomial $u^N(x)$ of degree $N+3$ interpolating the data

$$\begin{aligned} u_j &= u(x_j), \qquad 2 \leqslant j \leqslant N-1, \\ u(\pm 1) &= u'(\pm 1) = u''(\pm 1) = 0, \end{aligned} \tag{2.18}$$

and the polynomial $\tilde{u}^N(x)$ of degree $N+1$ interpolating the data

$$\begin{aligned} u_j &= u(x_j), \qquad 2 \leqslant j \leqslant N-1, \\ u(\pm 1) &= u'(\pm 1) = 0. \end{aligned} \tag{2.19}$$

It is shown in Huang and Sloan [4] that $u^N(x)$ may be expressed as

$$u^N(x) = \sum_{k=2}^{N-1} u_k \tilde{h}_k(x), \tag{2.20}$$

where

$$\tilde{h}(x) = \frac{(1-x^2)^2}{(1-x_k^2)^2} l_k(x) \tag{2.21}$$

and $\tilde{u}^N(x)$ has the form

$$\tilde{u}^N(x) = \sum_{k=2}^{N-1} u_k h_k(x), \tag{2.22}$$

where $h_k(x)$, $2 \leqslant k \leqslant N-1$, are defined by (2.8). The pseudospectral approximation to (2.16)–(2.17) is defined by the collocation equations

$$\frac{d^6 u^N}{dx^6}(x_j) = s\frac{d^4 \tilde{u}^N}{dx^4}(x_j), \qquad 2 \leqslant j \leqslant N-1. \tag{2.23}$$

If we denote

$$\begin{aligned} A_{j-1,k-1} &= \frac{d^6 \tilde{h}_k}{dx^6}(x_j), \\ B_{j-1,k-1} &= \frac{d^4 h_k}{dx^4}(x_j), \qquad 2 \leqslant k, j \leqslant N-1, \end{aligned} \tag{2.24}$$

and $\mathbf{u} = [u_2, ..., u_{N-1}]^{\mathrm{T}}$, we obtain the generalised matrix eigenvalue problem

$$A\mathbf{u} = sB\mathbf{u}, \tag{2.25}$$

or the scaled eigenvalue problem,

$$DA\mathbf{u} = sDB\mathbf{u}, \tag{2.26}$$

corresponding to (2.23). The numerical results for this example will be presented in Section 4.

From the two examples above it can be seen that the key points in our application of the pseudospectral method to eigenvalue problems are the proper choice of interpolation polynomials and the use of a different interpolating polynomial for each side of the differential equation. This approximation is extremely simple to use for linear eigenvalue problems and even for nonlinear ones (nonlinear in u or s) with Dirichlet boundary conditions. The elimination of spurious eigenvalues derives from the fact that the matrix B in (2.14) or (2.25) is definite. The definite property of B is necessary, but not sufficient, for the elimination of spurious eigenvalues. The elimination is associated with the proper choice of interpolating polynomial $\tilde{u}^N$. It is readily shown that if the same interpolating polynomial is used on each side of the differential equation then the matrix B is nearly singular. This near-singularity is the source of the spurious eigenvalues.

3. CALCULATION OF DIFFERENTIATION MATRICES

Before we report numerical results for the application of the pseudospectral method to eigenvalue problems, we consider how to calculate the derivatives of u^N (u^N is a certain interpolating polynomial) at collocation points, or equivalently, how to calculate differentiation matrices such as A and B in Section 2.

From the discussion in Section 2, we can consider the general polynomial $u^N(x)$ of degree $N+l_n+r_n-1$, interpolating the data

$$\begin{aligned} u_j &= u(x_j), && 2 \leqslant j \leqslant N-1 \\ u_1^{(v)} &= u^{(v)}(-1) = 0, && 0 \leqslant v \leqslant l_n, \\ u_N^{(v)} &= u^{(v)}(1) = 0, && 0 \leqslant v \leqslant r_n, \end{aligned} \tag{3.1}$$

where $l_n \geqslant 0$ and $r_n \geqslant 0$ are certain integers. As shown in Huang and Sloan [4], $u^N(x)$ may be expressed as

$$u^N(x) = \sum_{k=2}^{N-1} u_k h_k(x), \tag{3.2}$$

where

$$h_k(x) = \frac{(1+x)^{l_n}}{(1+x_k)^{l_n}} \cdot \frac{(1-x)^{r_n}}{(1-x_k)^{r_n}} l_k(x). \tag{3.3}$$

3.1. *The Fast Fourier Transform* (FFT) *Method*

When only a few eigenvalues of (2.14) and (2.25), or (2.15) and (2.26), are required iterative methods of solution are preferable. It is not necessary to form the complete matrices A and B. In this case the values $(d^M u^N/dx^M)(x_j)$, $2 \leqslant j \leqslant N-1$, are needed for some specified integer M. If $(x_j)_{j=1}^N$ are chosen to be Chebyshev collocation points $((x_j)_{j=2}^{N-1}$ are the zeros of $P_{N-2}^{(-1/2,-1/2)}(x))$ or Chebyshev–Lobatto points $((x_j)_{j=2}^{N-1}$ are the zeros of $P_{N-2}^{(1/2,1/2)}(x))$, the FFT method can be applied.

To apply the FFT to calculate $(d^M u^N/dx^M)(x_j)$, $2 \leqslant j \leqslant N-1$, we first let

$$v_k = \frac{u_k}{(1+x_k)^{l_n}(1-x_k)^{r_n}}, \qquad 2 \leqslant k \leqslant N-1, \tag{3.4}$$

and

$$v(x) = \sum_{k=2}^{N-1} v_k l_k(x), \tag{3.5}$$

where $v(x)$ is of degree $N-1$. Then $u^N(x)$ can be rewritten as

$$u^N(x) = (1+x)^{l_n}(1-x)^{r_n} v(x). \tag{3.6}$$

It is known (see, for example, Canuto *et al.* [1]) that the FFT can be employed in the evaluation of $(d^m v/dx^m)(x_j)$, $2 \leqslant j \leqslant N-1$, $0 \leqslant m \leqslant M$. Then using (3.6) it is easy to calculate $(d^M u^N/dx^M)(x_j)$, $2 \leqslant j \leqslant N-1$.

3.2. *The Method Using the Differentiation Matrices of Lagrange Interpolating Polynomials*

In many cases the formation of $(d^M u^N/dx^M)(x_j)$, $2 \leqslant j \leqslant N-1$, by matrix multiplication is preferable to the FFT method (see Solomonoff and Turkel [7]). Multiplying by a matrix has the advantage that it is more flexible and it is vectorisable. For example, both the location and the number of the collocation points are arbitrary. In this case we need to calculate the differentiation matrix A which is defined by

$$A_{jk} = \frac{d^M h_k}{dx^M}(x_j), \qquad 2 \leqslant j, k \leqslant N-1. \tag{3.7}$$

Referring to the expressions (3.3) for $h_k(x)$, $2 \leqslant k \leqslant N-1$, it is obvious that the key point in calculating A is the evaluation of $(d^m l_k/dx^m)(x_j)$, $2 \leqslant j, k \leqslant N-1$, and $0 \leqslant m \leqslant M$.

Now we are in a position to discuss how to calculate $(d^m l_k/dx^m)(x_j)$.

Define

$$a_k = \prod_{\substack{i=1 \\ i \neq k}}^{N} (x_k - x_i), \qquad 1 \leqslant k \leqslant N, \tag{3.8}$$

$$b_k = \sum_{\substack{i=1 \\ i \neq k}}^{N} \frac{1}{x_k - x_i}, \qquad 1 \leqslant k \leqslant N, \tag{3.9}$$

and

$$q_{jk}^{(m)}=\frac{d^m l_k}{dx^m}(x_j), \qquad 1\leqslant j,k\leqslant N, \quad m=0,1,2,\ldots. \tag{3.10}$$

LEMMA 3.1. *For $m\geqslant 1$, $q_{jk}^{(m)}$ is defined by*

$$q_{jj}^{(m)}=(-1)^{m-1}(m-1)!\sum_{l=0}^{m-1}\frac{(-1)^l}{l!}q_{jj}^{(l)}\times\left[\sum_{\substack{i=1\\i\neq j}}^{N}\frac{1}{(x_j-x_i)^{m-l}}\right],$$
$$q_{jk}^{(m)}=\frac{a_j(-1)^{m-1}m!}{a_k}\sum_{l=0}^{m-1}\frac{(-1)^l}{l!}\cdot\frac{q_{jj}^{(l)}}{(x_j-x_k)^{m-l}}, \quad j\neq k, \quad 1\leqslant j,k\leqslant N. \tag{3.11}$$

Proof. First let us prove the first relation of (3.11). From (2.2), we have

$$l_k(x)=\frac{1}{a_k}\prod_{\substack{i=1\\i\neq k}}^{N}(x-x_i)$$

and, therefore,

$$\frac{dl_k(x)}{dx}=l_k(x)\sum_{\substack{i=1\\i\neq k}}^{N}\frac{1}{x-x_i}. \tag{3.12}$$

From (3.12) and using the symbol $C_r^s=r!/(r-s)!s!$, we have

$$\begin{aligned}\frac{d^m l_k(x)}{dx^m}&=\sum_{l=0}^{m-1}C_{m-1}^l\frac{d^l l_k(x)}{dx^l}\sum_{\substack{i=1\\i\neq k}}^{N}\frac{d^{m-1-l}}{dx^{m-1-l}}\left(\frac{1}{x-x_i}\right)\\&=\sum_{l=0}^{m-1}C_{m-1}^l\frac{d^l l_k(x)}{dx^l}\sum_{\substack{i=1\\i\neq k}}^{N}\frac{(-1)^{m-1-l}(m-1-l)!}{(x-x_i)^{m-l}}\\&=\sum_{l=0}^{m-1}\frac{(-1)^{m-1-l}(m-1)!}{l!}\frac{d^l l_k(x)}{dx^l}\sum_{\substack{i=1\\i\neq k}}^{N}\frac{1}{(x-x_i)^{m-l}}.\end{aligned}$$

Putting $k=j$ and $x=x_j$ in the above equation we obtain the first relation of (3.11).

For the case $j\neq k$, we have

$$\begin{aligned}\frac{d^m l_k(x)}{dx^m}&=\frac{1}{a_k}\frac{d^m}{dx^m}\left[(x-x_j)\prod_{\substack{i=1\\i\neq k,j}}^{N}(x-x_i)\right]\\&=\frac{a_j}{a_k}\frac{d^m}{dx^m}\left[(x-x_j)\frac{l_j(x)}{x-x_k}\right]\\&=\frac{a_j}{a_k}\sum_{l=0}^{m}C_m^l\frac{d^l(x-x_j)}{dx^l}\frac{d^{m-l}}{dx^{m-l}}\left[\frac{l_j(x)}{x-x_k}\right]\\&=\frac{a_j}{a_k}\left\{(x-x_j)\frac{d^m}{dx^m}\left[\frac{l_j(x)}{x-x_k}\right]+m\frac{d^{m-1}}{dx^{m-1}}\left[\frac{l_j(x)}{x-x_k}\right]\right\}\\&=\frac{a_j}{a_k}\left\{(x-x_j)\frac{d^m}{dx^m}\left[\frac{l_j(x)}{x-x_k}\right]+m\sum_{l=0}^{m-1}\frac{(-1)^{m-1-l}(m-1)!}{l!}\frac{d^l l_j(x)}{dx^l}\cdot\frac{1}{(x-x_k)^{m-l}}\right\}.\end{aligned}$$

Putting $x=x_j$ in the equation above we obtain the second relation of (3.11).

After some algebraic manipulation we obtain from (3.11) the relations

$$q_{jj}^{(m)}=b_j q_{jj}^{(m-1)}-\sum_{\substack{i=1\\i\neq j}}^{N}\frac{a_i}{a_j(x_j-x_i)}q_{ji}^{(m-1)},$$
$$q_{jk}^{(m)}=\frac{a_j m}{a_k(x_j-x_k)}q_{jj}^{(m-1)}-\frac{m}{(x_j-x_k)}q_{jk}^{(m-1)}, \quad j\neq k,\ 1\leqslant j,k\leqslant N. \tag{3.13}$$

If we define

$$\tilde{q}_{jk}^{(m)}=cq_{jk}^{(m)}a_k, \qquad 1\leqslant j,k\leqslant N, \quad m=0,1,\ldots, \tag{3.14}$$

where $c\neq 0$ is a certain scaling factor, then (3.13) reads

$$\tilde{q}_{jj}^{(m)}=b_j\tilde{q}_{jj}^{(m-1)}-\sum_{\substack{i=1\\i\neq j}}^{N}\frac{\tilde{q}_{ji}^{(m-1)}}{(x_j-x_i)},$$
$$\tilde{q}_{jk}^{(m)}=\frac{m}{(x_j-x_k)}[\tilde{q}_{jj}^{(m-1)}-\tilde{q}_{jk}^{(m-1)}], \quad 1\leqslant j,k\leqslant N, \quad j\neq k, \quad m=1,2,\ldots. \tag{3.15}$$

For $m=0$ we have

$$\tilde{q}_{jk}^{(0)}=c\delta_{jk}a_k, \qquad 1\leqslant j,k\leqslant N. \tag{3.16}$$

$2N^2$ (multiplication or division) operations are required to compute a_k and b_k. Given a_k and b_k, $5N^2$ operations are required to find $\tilde{q}_{jk}^{(m)}$ from $\tilde{q}_{jk}^{(m-1)}$ for $m\geqslant 1$. Then $(5M+3)N^2$ operations are required to construct matrix $q_{jk}^{(M)}$ and about $(6M+3)N^2$ operations are needed to construct $q_{jk}^{(m)}$, $m=0,1,\ldots,M$.

4. NUMERICAL EXAMPLES

In this section two fourth-order and two sixth-order differential eigenvalue problems are solved using the

TABLE I

First Two Eigenvalues for Example 1 ($R=0$) Generated by the Pseudospectral Method with Chebyshev–Lobatto Points and Those Obtained in [2] Using Their Modified Chebyshev–Tau Method

	[2]		Present	
$N+1$	$s_1^{(N)}$	$s_2^{(N)}$	$s_1^{(N)}$	$s_2^{(N)}$
9	−9.8 700602	−20. 295078	−9.8 70467563383136	−20.1 2878236448789
14	−9.869604 5	−20.1. 290730	−9.86960440 2075734	−20.19072 797396664
19	−9.869604 31	−20.190728 6	−9.8696044010 90845	−20.1907285564 3102
24	−9.8696044	−20.190728	−9.8696044010 91274	−20.190728556426 84
29	−9.869604 7	−20.19072 9	−9.8696044010 93581	−20.19072855642 888
34	−9.8696 151	−20.19072 9	−9.8696044010 95919	−20.190728556426 58

Note. Exact eigenvalues: $s_1=-9.869604401089359$; $s_2=-20.19072855642663$.

pseudospectral method of Section 2. The eigenvalues obtained are compared to exact or accepted eigenvalues and to those obtained by other authors. The differentiation matrices are calculated using the method described in Section 3.2. The resulting generalised matrix eigenvalue problems are scaled by the diagonal preconditioners which are described in the Appendix and the scaled generalised matrix eigenvalue problems are solved using the routines F02BJF (for real matrices) or F02GJF (for complex matrices) from the NAG library on a VAX 8650 computer. All calculations were done using double precision (64-bit) arithmetic.

EXAMPLE 1 (A fourth-order eigenvalue problem with a third derivative term). Consider the eigenvalue problem used as Example 1 in Section 2,

$$u'''' + Ru''' = su'', \qquad -1 < x < 1, \qquad u(\pm 1) = u'(\pm 1) = 0. \tag{4.1}$$

The discretisations of this problem by the spectral tau method when $R=0$ and $R=4$ are shown in Gardner *et al.* [2] to have spurious eigenvalues.

Results using Chebyshev–Lobatto collocation points for the solution of problem (4.1) when $R=0$ and $R=4$ are given in Tables I and II, respectively. Tables I and II also contain the results obtained by Gardner *et al.* [2], using their modified Chebyshev–tau method and using single precision (66-bit) arithmetic on a Cray-2 supercomputer. The comparison can be done for polynomials of the same degree in the pseudospectral approximation as in the spectral tau method. The results in Tables I and II show that the discretisation by the pseudospectral method has no spurious eigenvalue and it is much more accurate than that of modified Chebyshev–tau method in [2].

Figures 1a, 1b, and 2 show $\log_{10}|s_1^{(N)}-s_1|$, $\log_{10}|s_2^{(N)}-s_2|$ for $R=0$, and $\log_{10}|s_1^{(N)}-s_1|$ for $R=4$, respectively, as functions of $(N+1)$ for the pseudospectral method with three sets of collocation points: Chebyshev, Legendre, and Chebyshev–Lobatto points ($(x_j)_{j=2}^{N-1}$ are zeros of $P_{N-2}^{(-1/2,-1/2)}(x)$, $P_{N-2}^{(0,0)}(x)$ and $P_{N-2}^{(1/2,1/2)}(x)$, respectively). It is obvious that the results obtained using these sets of collocation points are very close. Here s_i is the exact value of the ith eigenvalue and $s_i^{(N)}$ is the value computed using the pseudospectral method based on polynomials (2.7) and (2.10).

TABLE II

First Two Eigenvalues for Example 1 ($R=4$) Generated by the Pseudospectral Method with Chebyshev–Lobatto Points and Those Obtained in [2] Using Their Modified Chebyshev–Tau Method

	[2]	Present
$N+1$	$s_{1,2}^{(N)}$	$s_{1,2}^{(N)}$
9	−17.9 45354 ± 9.4 908166*i*	−17.9 2538900630421 ± 9.4 63513830784185*i*
14	−17.91292 4 ± 9.458 3902*i*	−17.912921 97234431 ± 9.45840 0313956346*i*
19	−17.91292 2 ± 9.4584014*i*	−17.9129218001 5564 ± 9.45840144300 3609*i*
24	−17.91292 2 ± 9.458401 5*i*	−17.91292180018 556 ± 9.458401443007 839*i*
29	−17.91292 2 ± 9.458401 9*i*	−17.91292180018 723 ± 9.45840144300 8758*i*
34	−17.91292 5 ± 9.45840 97*i*	−17.91292180018 810 ± 9.45840144300 9625*i*

Note. Exact eigenvalues: $s_{1,2}=-17.91292180018440 \pm 9.458401443007244i$. Spaces in numbers show the extent of agreement with exact values.

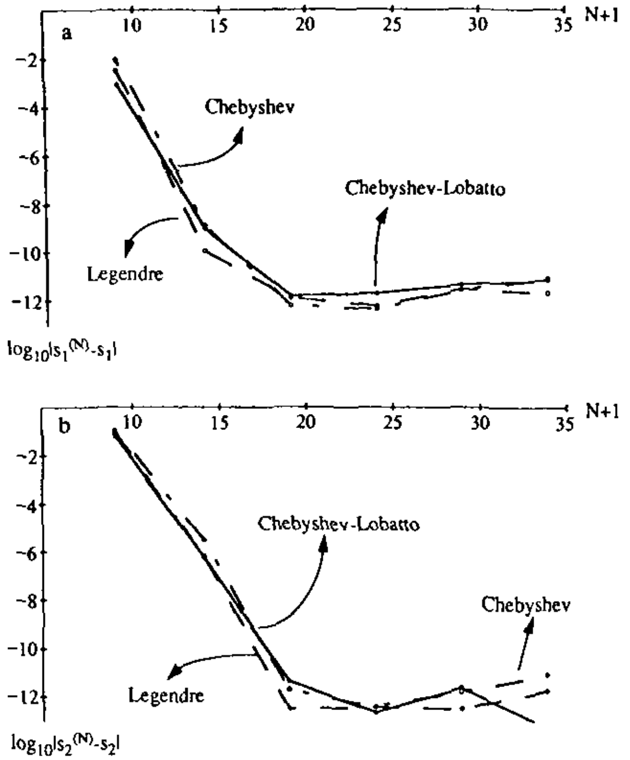

FIG. 1. (a) $\log_{10}|s_1^{(N)}-s_1|$ as a function of $N+1$ for problem (4.1) with $R=0$. (b) $\log_{10}|s_2^{(N)}-s_2|$ as a function of $N+1$ for problem (4.1) with $R=0$.

Example 2. (A sixth-order eigenvalue problem). Consider the eigenvalue problem used as Example 2 in Section 2,

$$\frac{d^6u}{dx^6}=s\frac{d^4u}{dx^4}, \qquad -1<x<1, \qquad (4.2)$$
$$u(\pm 1)=u'(\pm 1)=u''(\pm 1)=0.$$

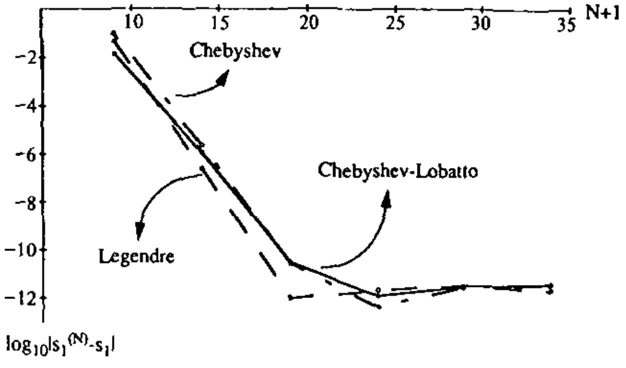

FIG. 2. $\log_{10}|s_1^{(N)}-s_1|$ as a function of $N+1$ for problem (4.1) with $R=4$.

TABLE III

First Two Eigenvalues for Example 2 Generated by the Pseudospectral Method with Chebyshev–Lobatto Points

$N+3$	$s_1^{(N)}$	$s_2^{(N)}$
8	−20. 27448865769315	−84.00000000000001
12	−2 1.44588544085100	−3 4.67499895206012
18	−20.190728556 37254	−33.2174619 8708751
22	−20.19072855642 984	−33.2174619142 7011
28	−20.1907285564 9336	−33.21746191426 238
32	−20.190728556 25974	−33.2174619142 5708
38	−20.19072855 950501	−33.217461914 30863

Note. Exact eigenvalues:
$s_1=-20.19072855642663$; $s_2\approx-33.21746191426837$.

Results obtained using Chebyshev–Lobatto points for (4.2) are given in Table III. Figures 3a, b show $\log_{10}|s_1^{(N)}-s_1|$ and $\log_{10}|s_2^{(N)}-s_2|$, respectively, as functions of $N+3$ for the pseudospectral method with three sets of collocation points: Chebyshev, Legendre and Chebyshev–Lobatto points. Here $s_i^{(N)}$ is the value of the ith eigenvalue computed using the pseudospectral method based on polynomials (2.20) and (2.22).

The results show that the pseudospectral method has very fast convergence and has no spurious eigenvalue. It is easy to see that the calculations suffer from roundoff error when $N+3\geqslant 22$. Nevertheless, the computed solutions agree with the exact solutions at least up to nine digits when $N+3\geqslant 18$.

Example 3. (The Orr–Sommerfeld stability equation for plane Poiseuille flow). The Orr–Sommerfeld stability equation for plane Poiseuille flow has been solved by a variety of methods, including the spectral tau method (see Orszag [6], Gardner *et al.* [2], McFadden *et al.* [5]). The equation results from assuming that a velocity disturbance of the form

$$V(x,y,t)=u(x)\exp[i\sigma(y-st)] \qquad (4.3)$$

perturbs the steady pressure-induced flow $U(x)=(1-x^2)$. between two infinite parallel plates located (in dimensionless variables) at $x=\pm 1$. The resulting linear stability equation is

$$[u''''-2\sigma^2u''+\sigma^4u]/(-i\sigma R)+[U(u''-\sigma^2u)-U''u]$$
$$=s(u''-\sigma^2u), \qquad -1<x<1, \qquad (4.4)$$

with boundary conditions

$$u(\pm 1)=u'(\pm 1)=0, \qquad (4.5)$$

where u is the amplitude of the velocity disturbance (defined in (4.3)) and R is the Reynolds number.

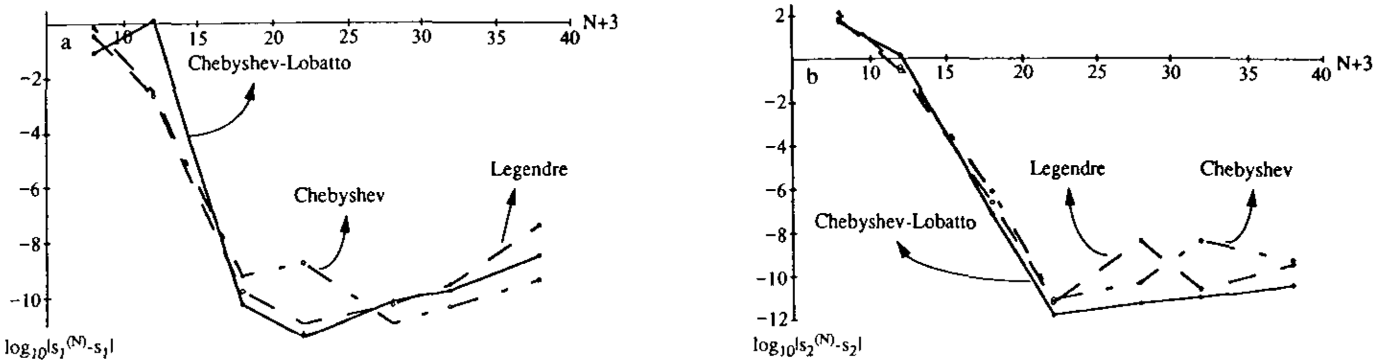

FIG. 3. (a) $\log_{10}|s_1^{(N)}-s_1|$ as a function of $N+3$ for problem (4.2). (b) $\log_{10}|s_2^{(N)}-s_2|$ as a function of $N+3$ for problem (4.2).

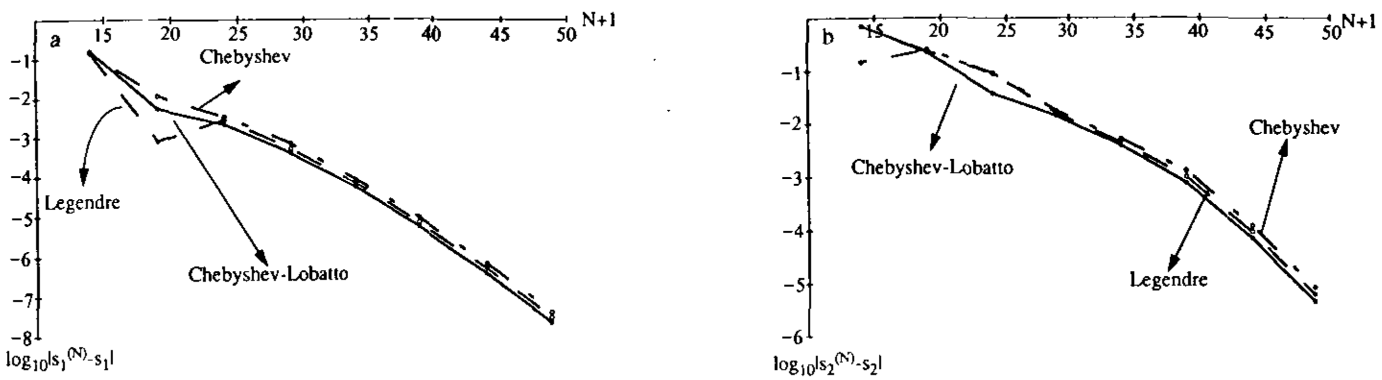

FIG. 4. (a) $\log_{10}|s_1^{(N)}-s_1|$ as a function of $N+1$ for Orr–Sommerfeld problem (4.4)–(4.5) with $\sigma=1$ and $R=10^4$. (b) $\log_{10}|s_2^{(N)}-s_2|$ as a function of $N+1$ for Orr–Sommerfeld problem (4.4)–(4.5) with $\sigma=1$ and $R=10^4$.

TABLE IV

First Two Eigenvalues for Example 3 ($\sigma=1.00$, $R=10{,}000$) Generated by the Pseudospectral Method with Chebyshev–Lobatto Points and Those Obtained in [2] Using Their Modified Chebyshev–Tau Method

$N+1$	[2] $s_1^{(N)}$	Present $s_1^{(N)}$	Present $s_2^{(N)}$
14	0. 52900096 + 0. 22074414i	0. 36841258081 + 0.0 6487680791i	0. 27408233596 + 0.03376347334i
19	0. 73111753 + 0.0 96973658i	0.237 09997536 + 0.00 441789200i	0. 72863350430 − 0.0 0009648265i
24	0.2 4033386 + 0.00 64426763i	0.23 563796411 + 0.00 151181136i	0. 87829954112 − 0.0 1682408694i
29	0.2375 7258 + 0.0037 438271i	0.237 20866786 + 0.003 32860850i	0.9 5300865710 − 0.0 2437870062i
34	0.2375 5789 + 0.0037 060033i	0.237 48476605 + 0.003 68618134i	0.964 53117558 − 0.03 066387996i
39	0.23752 741 + 0.0037 419091i	0.23752 489827 + 0.00373 265586i	0.96 365408185 − 0.03 499614001i
44	*	0.237526 75297 + 0.003739 21305i	0.964 71323576 − 0.0351 2106936i
49	*	0.237526 51907 + 0.00373967 171i	0.96463 578946 − 0.03516 32746i

Note. Exact eigenvalues: $s_1=0.23752648882+0.00373967062i$; $s_2=0.9646309154-0.0351672775i$.

TABLE V

First Two Eigenvalues for Example 4 Generated by the Pseudospectral Method with Chebyshev–Lobatto Points

$N+3$	$s_1^{(N)}$	$s_2^{(N)}$
8	−23 1.53643345040102	−3359.999999999999
12	−237. 6992290474482	−7 74.6693699641236
18	−237.72106753 20906	−769.96348 24081218
22	−237.721067531 0667	−769.9634832419 836
28	−237.721067531 6501	−769.96348324 09488
32	−237.7210675 278370	−769.963483241 2687
38	−237.7210675 205215	−769.9634832 398821

Note. Exact eigenvalues:
$s_1 = -237.72106753111$; $s_2 = -769.96348324190$.

Problem (4.4)–(4.5) was solved for $\sigma = 1.00$ and $R = 10{,}000$ using the pseudospectral method. Here the pseudospectral method is based on approximations to the left-hand side and right-hand side of (4.4) using polynomials (2.7) and (2.10), respectively. The results for the first two eigenvalues (arranged in order of decreasing imaginary part) using Chebyshev–Lobatto points are presented in Table IV. Convergence to the first eigenvalue can be compared with that obtained by Gardner *et al.* [2], using their modified Chebyshev–tau method. Our results for the second eigenvalue are comparable in accuracy to those obtained by McFadden *et al.* [5] (with a truncated Chebyshev series of degree 50 they obtained the approximation $0.96462731 - 0.03516958i$).

From Table IV it can be seen that the pseudospectral approximation to problem (4.4)–(4.5) for $\sigma = 1.00$ and $R = 10{,}000$ has no spurious eigenvalue when $N+1 \geqslant 19$ and is comparable in accuracy to the tau method. It is also worth noting that both the pseudospectral method and the tau method converge more slowly for this example than for Examples 1 and 2. The reason for this will be the singular nature of the eigenvalue problem caused by the large value of R.

Figures 4a, b show $\log_{10} |s_1^{(N)} - s_1|$ and $\log_{10} |s_2^{(N)} - s_2|$, respectively, as functions of $(N+1)$ for the pseudospectral method with three sets of collocation points.

EXAMPLE 4. (Another sixth-order eigenvalue problem). Another sixth-order eigenvalue problem for which it was shown in McFadden *et al.* [5] that the tau approximation has spurious eigenvalues is

$$\frac{d^6 u}{dx^6} = -s\,\frac{d^2 u}{dx^2}, \qquad -1 < x < 1,$$
$$u(\pm 1) = u'(\pm 1) = u''(\pm 1) = 0. \tag{4.6}$$

Results for (4.6) generated by a pseudospectral method like that used in Example 2 using Chebyshev–Lobatto points are given in Table V. Figures 5a, b show $\log_{10} |s_1^{(N)} - s_1|$ and $\log_{10} |s_2^{(N)} - s_2|$, respectively, as functions of $N+3$ for three sets of collocation points. The results for (4.6) have nearly the same behaviour as those for Example 2 (4.2). The computed solutions agree with the exact ones at least up to eight digits when $N+3 \geqslant 18$.

5. CONCLUSIONS AND COMMENTS

We have presented a pseudospectral method for the numerical solution of linear, differential eigenvalue problems which, unlike the commonly used spectral tau method, does not generate spurious eigenvalues. The method is flexible in the sense that it deals with any homogeneous Dirichlet boundary conditions and with arbitrary interior collocation points. A useful algorithm is also presented for the generation of pseudospectral differentiation matrices at general collocation points.

Computations performed on four model problems indicate that the pseudospectral method is at least as accurate as the modified Chebyshev tau method which was recently proposed by Gardner *et al.* [2]. This latter method

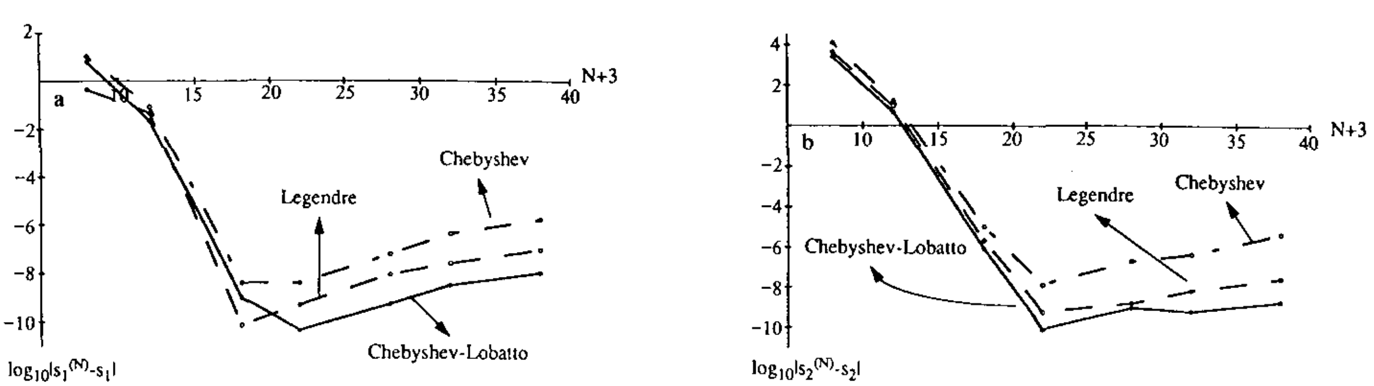

FIG. 5. (a) $\log_{10} |s_1^{(N)} - s_1|$ as a function of $N+3$ for problem (4.6). (b) $\log_{10} |s_2^{(N)} - s_2|$ as a function of $N+3$ for problem (4.6).

is a modified spectral tau method which eliminates spurious eigenvalues, and it is comparable in accuracy to the standard spectral tau method.

The method which we have presented has the usual advantages associated with pseudospectral methods: it is easy to implement and it deals in a straightforward manner with variable coefficients. We feel that it should prove useful in flow stability calculations. Since the method does not generate spurious (unstable) eigenvalues it may also prove useful in time-dependent calculations.

APPENDIX: PRECONDITIONER FOR PSEUDOSPECTRAL DIFFERENTIATION MATRICES

To simplify the presentation of the preconditioner D which is introduced in (2.15), suppose the matrix A in (2.14) is the pseudospectral approximation to the operator

$$L^{(n)}u = \frac{d^n u}{dx^n} \quad \text{in} \quad [-1, 1], \tag{A.1}$$

subject to the boundary conditions

$$\begin{aligned} u^{(\mu)}(-1) &= 0, \qquad 0 \leqslant \mu \leqslant l_n, \\ u^{(\nu)}(+1) &= 0, \qquad 0 \leqslant \nu \leqslant r_n, \end{aligned} \tag{A.2}$$

where $n \geqslant 1$, $l_n \geqslant -1$, and $r_n \geqslant -1$ are integers such that $l_n + r_n + 2 = n$ (referring to Example 1, the situation above would arise if we set R to zero and assumed the values $n = 4$, $l_n = r_n = 1$). As shown in [4], the interpolating polynomial of degree at most $N + n - 3$ which satisfies conditions (A.2) and assumes values $u_k = u(x_k)$ at the interior collocation points in the set (2.1) is given by

$$u^{N+n-3}(x) = \sum_{k=2}^{N-1} u_k h_k(x). \tag{A.3}$$

The pseudospectral differentiation matrix $L_{sp}^{(n)}$ is given by

$$(L_{sp}^{(n)})_{jk} = \frac{d^n h_k}{dx^n}(x_j), \qquad 2 \leqslant k, j \leqslant N-1, \tag{A.4}$$

where

$$h_k(x) = \frac{(1+x)^{l_n+1}(1-x)^{r_n+1}}{(1+x_k)^{l_n+1}(1-x_k)^{r_n+1}} \cdot \frac{\pi(x)}{\pi'(x_k)(x-x_k)}, \qquad 2 \leqslant k \leqslant N-1, \tag{A.5}$$

and

$$\pi(x) = \prod_{i=2}^{N-1} (x - x_i). \tag{A.6}$$

Define the condition number $\kappa(L_{sp}^{(n)})$ as

$$\kappa(L_{sp}^{(n)}) = \frac{|\lambda|_{\max}}{|\lambda|_{\min}}, \tag{A.7}$$

where λ is the eigenvalue of $L_{sp}^{(n)}$. It is known that $\kappa(L_{sp}^{(1)}) = O(N^2)$ and $\kappa(L_{sp}^{(2)}) = O(N^4)$ as $N \to \infty$ for general collocation points. We define a preconditioner for $L_{sp}^{(n)}$ wich reduces the condition number to $O(N^n)$ as $N \to \infty$. Let

$$D = \operatorname{diag}((1+x_k)^{l_n+1}(1-x_k)^{r_n+1}, 2 \leqslant k \leqslant N-1) \tag{A.8}$$

and consider the eigenvalues of matrix $DL_{sp}^{(n)}$. It is obvious that the eigenproblem

$$DL_{sp}^{(n)} v = \tau v, \tag{A.9}$$

where $v = [v_2, ..., v_{N-1}]^T$, can be rewritten as

$$\begin{aligned} &(1+x)^{l_n+1}(1-x)^{r_n+1} \frac{d^n}{dx^n} v^{N+n-3}\Big|_{x_k} \\ &\quad = \tau v^{N+n-3}(x_k), \qquad 2 \leqslant k \leqslant N-1, \end{aligned} \tag{A.10}$$

where the polynomial v^{N+n-3} of degree at most $N+n-3$ is given by

$$v^{N+n-3}(x) = \sum_{k=2}^{N-1} v_k h_k(x). \tag{A.11}$$

Noting that both sides of (A.10) are polynomials of degree at most $N + n - 3$, we have

$$\begin{aligned} &(1+x)^{l_n+1}(1-x)^{r_n+1} \frac{d^n}{dx^n} v^{N+n-3}(x) \\ &\quad = \tau v^{N+n-3}(x) + \pi(x) \sum_{i=0}^{n-1} a_i x^i \qquad \text{in } [-1, 1]. \end{aligned} \tag{A.12}$$

We note that $v^{N+n-3}(x)$ and the left-hand side of (A.12) satisfy the boundary conditions (A.2). Hence the polynomial $\pi(x) \sum_{i=0}^{n-1} a_i x^i$ should satisfy the conditions (A.2); therefore, because $\pi(\pm 1) \neq 0$, $\sum_{i=0}^{n-1} a_i x^i$ should satisfy the conditions (A.2) and this implies that $a_i = 0$, $i = 0, ..., n-1$. Thus, from (A.11) and (A.5) we have proved the following theorem.

THEOREM A.1. *The eigenvalues of the preconditioned pseudospectral differentiation matrix $DL_{sp}^{(n)}$ are determined by the eigenproblem*

$$\begin{aligned} &\frac{d^n}{dx^n}[(1+x)^{l_n+1}(1-x)^{r_n+1} p(x)] \\ &\quad = \tau p(x), \qquad p(x) \in \mathbf{P}_{N-3}. \end{aligned} \tag{A.13}$$

Therefore, the eigenvalues of $DL_{sp}^{(n)}$ are independent of the choice of collocation points.

Remark. Note that the preconditioned matrix $DL_{sp}^{(n)}$ is just the pseudospectral differentiation matrix of operator $(1+x)^{l_n+1}(1-x)^{r_n+1}(d^n/dx^n)$ subject to (A.2). Then the theorem implies that the eigenvalues of the differentiation matrix are independent of the choice of collocation points!

THEOREM A.2. *The eigenvalues of $DL_{sp}^{(n)}$ are real and given by*

$$\tau_k=(-1)^{r_n+1}\prod_{i=1}^{n}(k+i), \qquad 0\leqslant k\leqslant N-3. \quad \text{(A.14)}$$

Therefore, the condition number of $DL_{sp}^{(n)}$ is

$$\kappa(DL_{sp}^{(n)})=\prod_{i=1}^{n}\frac{N-3+i}{i}=O(N^n). \qquad \text{(A.15)}$$

The proof is fairly straightforward.

It may be shown that a similar reduction in condition number may be achieved if the operator $L^{(n)}u$ in (A.1) is replaced by

$$\mathscr{L}^{(n)}u=\sum_{i=0}^{n}c_i\frac{d^iu}{dx^i}, \qquad \text{(A.16)}$$

where c_i $(i=0, 1, ..., n)$ are constants, $c_n\neq 0$, and $u(x)$ satisfies (A.2). The preconditioner D is also effective for a wide class of linear differential operators involving smooth, variable coefficients.

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